

ON CUSPIDAL REPRESENTATIONS OF GENERAL LINEAR GROUPS OVER DISCRETE VALUATION RINGS

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ABSTRACT. We define a new notion of cuspidality for representations of GL_n over a finite quotient \mathfrak{o}_k of the ring of integers \mathfrak{o} of a non-Archimedean local field F using geometric and infinitesimal induction functors, which involve automorphism groups G_λ of torsion \mathfrak{o} -modules. When n is a prime, we show that this notion of cuspidality is equivalent to strong cuspidality, which arises in the construction of supercuspidal representations of $\mathrm{GL}_n(F)$. We show that strongly cuspidal representations share many features of cuspidal representations of finite general linear groups. In the function field case, we show that the construction of the representations of $\mathrm{GL}_n(\mathfrak{o}_k)$ for $k \geq 2$ for all n is equivalent to the construction of the representations of all the groups G_λ . A functional equation for zeta functions for representations of $\mathrm{GL}_n(\mathfrak{o}_k)$ is established for representations which are not contained in an infinitesimally induced representation. All the cuspidal representations for $\mathrm{GL}_4(\mathfrak{o}_2)$ are constructed. Not all these representations are strongly cuspidal.

1. INTRODUCTION

The irreducible characters of $\mathrm{GL}_n(\mathbf{F}_q)$ were computed by J. A. Green in 1955 [Gre55]. In Green's work, parabolic induction was used to construct many irreducible characters of $\mathrm{GL}_n(\mathbf{F}_q)$ from irreducible characters of smaller general linear groups over \mathbf{F}_q . The representations which could not be obtained in this way, known as *cuspidal representations*, were shown to be in canonical bijective correspondence with Galois orbits of norm-primitive characters of $\mathbf{F}_{q^n}^\times$ (these are characters which do not factor through the norm map $\mathbf{F}_{q^n}^\times \rightarrow \mathbf{F}_{q^d}^\times$ for any proper factor d of n).

Let F be a non-Archimedean local field with ring of integers \mathfrak{o} . Let \mathfrak{p} be the maximal ideal in \mathfrak{o} , and $\mathfrak{o}_k = \mathfrak{o}/\mathfrak{p}^k$ for $k \geq 1$. Thus \mathfrak{o}_1 is a finite field, the residue field of F , which we take to be \mathbf{F}_q . In contrast with $\mathrm{GL}_n(\mathbf{F}_q)$, not much is known in general about the representation theory of $\mathrm{GL}_n(\mathfrak{o}_k)$. Unlike general linear groups over fields, for which conjugacy classes are parameterized by Jordan canonical forms, the classification of conjugacy classes in $\mathrm{GL}_n(\mathfrak{o}_k)$ for all n and any $k \geq 2$ contains the matrix pair problem [Nag78, Section 4], which is a wild classification problem.

The representations of $\mathrm{GL}_n(\mathfrak{o})$ received considerable attention after supercuspidal representations of $\mathrm{GL}_n(F)$ were constructed by induction from a compact-modulo-center subgroup [Shi68, How77, Kut78]. A class of representations (*représentations très cuspidales*) of the

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maximal compact-modulo-center subgroups which give rise to irreducible supercuspidal representations of $\mathrm{GL}_n(F)$ were identified by Carayol [Car84]. When the maximal compact subgroup modulo center in question is $F^\times \mathrm{GL}_n(\mathfrak{o})$, the restrictions of these representations to $\mathrm{GL}_n(\mathfrak{o})$ correspond to what we call *strongly cuspidal representations* of $\mathrm{GL}_n(\mathfrak{o}_k)$ for some k (Definition 4.1). Carayol used these representations to construct all the supercuspidal representations of $\mathrm{GL}_n(F)$ when n is prime. The classification of supercuspidal representations of $\mathrm{GL}_n(F)$ for n arbitrary was completed by Bushnell and Kutzko [BK93]. Recently, Paskunas [Pas05] proved that given an irreducible supercuspidal representation π of $\mathrm{GL}_n(F)$, there exists a unique (up to isomorphism) irreducible representation τ of $\mathrm{GL}_n(\mathfrak{o})$, such that τ is a type for the Bernstein component of π . Hence representations of $\mathrm{GL}_n(\mathfrak{o})$ occur naturally in the representation theory of $\mathrm{GL}_n(F)$. Nevertheless, with respect to $\mathrm{GL}_n(\mathfrak{o})$, since the general representation theory is unmanageably complicated, only those very special representations that are needed to understand the representations of the p -adic group itself have been considered.

In this article, we take the point of view that the representation theory of $\mathrm{GL}_n(\mathfrak{o})$ is interesting in its own right, and while extremely complicated, does display a certain structure. To this end, a new definition of cuspidality is introduced for representations of $\mathrm{GL}_n(\mathfrak{o}_k)$. This definition is closer in spirit to the characterization in [Gre55] of cuspidal representations as those which do not occur in representations obtained by parabolic induction. More specifically, let Λ denote the set of all partitions of all positive integers. The isomorphism classes of finitely generated torsion \mathfrak{o} -modules are parameterized by Λ . For any \mathfrak{o} -module $\mathfrak{o}_\lambda = \bigoplus_{i=1}^m \mathfrak{o}_{\lambda_i}$ of type $\lambda = (\lambda_1, \dots, \lambda_m) \in \Lambda$, let $G_\lambda = G_{\lambda, F}$ stand for its automorphism group. Thus, for example, $G_{k^n} = \mathrm{GL}_n(\mathfrak{o}_k)$. Say that $\lambda \leq \mu$ if \mathfrak{o}_λ can be embedded in \mathfrak{o}_μ . We call an irreducible representation of G_{k^n} *cuspidal* (see Definition 4.2) if it cannot be constructed from lower building blocks. By lower building blocks we mean the representations of G_λ , where $\lambda < k^n$. These automorphism groups play the role of Levi components of proper parabolic subgroups of $\mathrm{GL}_n(\mathfrak{o}_1)$. Representations of G_{k^n} are constructed from those of G_λ using *infinitesimal* and *geometric* induction (Section 3). Our first result, which is proved in Section 4.2, compares cuspidality with strong cuspidality.

Theorem A. *Every strongly cuspidal representation is cuspidal. When n is prime every cuspidal representation is strongly cuspidal.*

When n is not prime, it is not true that every cuspidal representation is strongly cuspidal. In Section 8, all the cuspidal representations of $\mathrm{GL}_4(\mathfrak{o}_2)$ are constructed. Among these are representations which are not strongly cuspidal.

The construction of strongly cuspidal representations is well-known [Shi68, Gér75, How77]. When n is prime, then by Theorem A, all cuspidal representations are obtained in this manner. Moreover, for all n , the strongly cuspidal representations have properties analogous to cuspidal representations of $\mathrm{GL}_n(\mathfrak{o}_1)$. Firstly, they can be parameterized in an analogous fashion. Suppose that E is an unramified extension of F of degree n , and \mathfrak{D} is the integral closure of \mathfrak{o} in E . Let \mathfrak{P} denote the maximal ideal in \mathfrak{D} and \mathfrak{D}_k denote the finite quotient ring $\mathfrak{D}/\mathfrak{P}^k$. For $k > 1$, a character $\mathfrak{D}_k^\times \rightarrow \mathbf{C}^\times$ is said to be *strongly primitive* if its restriction to $\ker(\mathfrak{D}_k^\times \rightarrow \mathfrak{D}_{k-1}^\times) \cong \mathfrak{D}_1$ does not factor through any proper subfield via the trace map. A character of \mathfrak{D}_1^\times is said to be strongly primitive if it is norm-primitive. In Section 5.3 we prove

Theorem B. *There is a canonical bijective correspondence between strongly cuspidal representations of G_{k^n} and $\text{Gal}(E/F)$ -orbits of strongly primitive characters of \mathfrak{D}_k^\times .*

The group of units \mathfrak{D}_k^\times is embedded in G_{k^n} since $\mathfrak{D}_k \simeq \mathfrak{o}_k^n$ as \mathfrak{o}_k -modules. An element of G_{k^n} is said to be regular elliptic if it is conjugate to an element of \mathfrak{D}_k^\times whose image in \mathfrak{D}_1^\times lies in no proper subfield. In section 5.2, we establish another property that strongly cuspidal representations share with cuspidal representations of $\text{GL}_n(\mathfrak{o}_1)$, which is that the correspondence of Theorem B is well-behaved with respect to character values on regular elliptic elements.

Theorem C. *Let ω be a strongly primitive character of \mathfrak{D}_k^\times and let Θ_ω be the corresponding strongly cuspidal character of G_{k^n} . Then for all regular elliptic elements $u \in \mathfrak{D}_k^\times \subset G_{k^n}$*

$$\Theta_\omega(u) = (-1)^{(n-1)k} \sum_{\gamma \in \text{Gal}(E/F)} \omega(\gamma u),$$

Moreover, Θ_ω vanishes on conjugacy classes which do not intersect $\mathfrak{D}_k^\times \cdot \text{Ker}\{G_{k^n} \rightarrow G_{[k/2]^n}\}$.

Remark. Theorems B and C are due to Green when $k = 1$. For $k > 1$, the ideas used in the proofs can be found in the existing literature on supercuspidal representations of $\text{GL}_n(F)$, the detailed account in Section 5 gives the complete picture, working entirely inside $\text{GL}_n(\mathfrak{o})$. In particular, Theorem C is deduced from [Gér75, Theorem 1]. It is closely related to the result obtained by Henniart in [Hen93, Section 3.7]. We also observe that in [Lu04] Lusztig gave a geometric construction of representations (in the function field case) which is likely to include the description of strongly cuspidal representations of G_{k^n} in terms of strongly primitive characters.

There already is evidence that the representation theory of a group such as G_λ can be studied by breaking up the problem into two parts. The first is to correctly define and understand the cuspidal representations. The second is to construct the remaining representations from cuspidal representations of G_μ with $\mu < \lambda$. This approach has been implemented successfully in [Onn07] for automorphism groups of modules of rank two. Theorems A, B and C provide further evidence of the validity of this approach when $\lambda = k^n$ and n is a prime.

The inevitability of the family of groups G_λ in the representation theory of G_{k^n} or even G_{2^n} can be seen from another perspective. In Section 6, we prove

Theorem D. *Let F be a local function field. Constructing the irreducible representations of the family of groups $\{G_{2^n, F} = \text{GL}_n(\mathfrak{o}_2) \mid n \in \mathbf{N}\}$ is equivalent to constructing the irreducible representations of the family $\{G_{\lambda, E} \mid \lambda \in \Lambda, E/F \text{ unramified extension}\}$.*

Finally, we point out a suggestive connection to the Macdonald correspondence which might admit a higher level incarnation as well. Macdonald has established a correspondence between irreducible representations of G_{1^n} and equivalence classes of n -dimensional tamely ramified representations of the Weil-Deligne group W'_F [Mac80]. One ingredient in this correspondence is a functional equation for the zeta function associated to G_{1^n} . It admits a straightforward generalization to G_{k^n} for $k > 1$. Let \hat{f} denote a properly normalized additive Fourier transform of $f \in \mathbf{C}(M_n(\mathfrak{o}_k))$ with respect to $\psi(\text{tr}(\cdot))$, where $\psi : \mathfrak{o}_k \rightarrow \mathbf{C}$ is an additive character which does not factor through \mathfrak{o}_{k-1} . Let $\mathcal{Z}(f, \rho) = \sum_{g \in G_{k^n}} \hat{f}(g) \rho(g) \in \text{End}_{\mathbf{C}}(V)$ where $f \in \mathbf{C}(M_n(\mathfrak{o}_k))$ and (ρ, V) is an irreducible representation of G_{k^n} . Denote by $\check{\rho}$ the contragredient representation of ρ . In Section 7, we prove

Theorem E. *If ρ is not contained in an infinitesimally induced representation (in particular if ρ is cuspidal), there exists a complex number $\varepsilon(\rho, \psi)$ and a such that*

$${}^t\mathcal{Z}(\hat{f}, \check{\rho}) = \varepsilon(\rho, \psi) \mathcal{Z}(f, \rho).$$

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2. NOTATIONS AND PRELIMINARIES

2.1. Automorphism groups. Let Λ denote the set of all partitions of all positive integers. Any $\lambda \in \Lambda$ can be written in the form $(\lambda_1^{r_1}, \dots, \lambda_l^{r_l})$, where $\lambda_1 > \dots > \lambda_l$ and r_1, \dots, r_l are positive integers. The sum $r_1 + \dots + r_l$ will be called the *length* of the partition, and λ_1 will be called the *height* of the partition.

Every finitely generated torsion \mathfrak{o} -module is of the form $\mathfrak{o}_\lambda = \mathfrak{o}_{\lambda_1}^{r_1} \oplus \dots \oplus \mathfrak{o}_{\lambda_l}^{r_l}$ for some $\lambda \in \Lambda$ of height no more than k . Consider the group $G_\lambda = \text{Aut}_{\mathfrak{o}}(\mathfrak{o}_\lambda)$. In particular, taking $\lambda = (k^n)$, we have $G_{k^n} = \text{GL}_n(\mathfrak{o}_k)$. When it is necessary to specify the underlying non-Archimedean local field F , the notation $G_{\lambda, F}$ will be used for G_λ .

Let N_r denote the kernel of the natural map $G_{k^n} \rightarrow G_{r^n}$. Then, if $r \geq k/2$, the map $M_n(\mathfrak{o}_{k-r}) \rightarrow N_r$ defined by $A \mapsto I + \varpi^r A$, is an isomorphism of groups (it is a bijection of sets for all $r < k$). This results in a short exact sequence

$$(2.1) \quad 0 \rightarrow M_n(\mathfrak{o}_{k-r}) \rightarrow G_{k^n} \rightarrow G_{r^n} \rightarrow 1,$$

for every $r \geq k/2$. In what follows, we identify $M_n(\mathfrak{o}_{k-r})$ with its image in G_{k^n} for $r \geq k/2$.

2.2. Similarity classes associated to representations. Assume that $r \geq k/2$. The action of G_{k^n} on its normal subgroup $M_n(\mathfrak{o}_{k-r})$ factors through $G_{(k-r)^n}$. In fact, this is just the usual action by similarity transformations

$$g \cdot A = gAg^{-1}, \quad g \in G_{(k-r)^n}, \quad A \in M_n(\mathfrak{o}_{k-r}).$$

It results in an action of $G_{(k-r)^n}$ on the set of all characters of $M_n(\mathfrak{o}_{k-r})$.

Now suppose that ρ is an irreducible representation of G_{k^n} on a vector space V . The restriction of ρ to $M_n(\mathfrak{o}_{k-r})$ gives rise to a decomposition $V = \oplus V_\chi$, where χ ranges over the set of characters of $M_n(\mathfrak{o}_{k-r})$. Clifford theory then tells us that the set of characters χ for which V_χ is non-trivial consists of a single orbit for the action of $G_{(k-r)^n}$ on the characters of $M_n(\mathfrak{o}_{k-r})$.

The group $M_n(\mathfrak{o}_{k-r})$ can be identified with its Pontryagin dual (as a $G_{(k-r)^n}$ -space). For this, pick an additive character ψ of $F \rightarrow \mathbf{C}^\times$ whose restriction to \mathfrak{o} is trivial, but whose restriction to \mathfrak{p}^{-1} is non-trivial. For each $A \in M_n(\mathfrak{o}_{k-r})$, define a character $\psi_A: M_n(\mathfrak{o}_{k-r}) \rightarrow \mathbf{C}^\times$ by $\psi_A(B) = \psi(\varpi^{r-k} \text{tr}(AB))$. The map $A \mapsto \psi_A$ identifies $M_n(\mathfrak{o}_{k-r})$ with its Pontryagin dual, and preserves the action of $G_{(k-r)^n}$.

Thus we associate, for each $r \geq k/2$, to each irreducible representation ρ of G_{k^n} , a similarity class $\Omega_{k-r}(\rho) \subset M_n(\mathfrak{o}_{k-r})$.

3. INDUCTION AND RESTRICTION FUNCTORS

This section introduces the functors that will play the role of parabolic induction and restriction in the context of $\mathrm{GL}_n(\mathfrak{o}_k)$. They were introduced in [Onn07, Section 2]. Geometric induction is an obvious analog of parabolic induction in the case of a field. Infinitesimal induction has no analog in that setting.

3.1. Geometric induction and restriction functors. Given a direct sum decomposition $\mathfrak{o}_k^n = \mathfrak{o}_k^{n_1} \oplus \mathfrak{o}_k^{n_2}$, define P_{n_1, n_2} to be the subgroup of G_{k^n} which preserves $\mathfrak{o}_k^{n_1}$. There is a natural surjection $\varphi: P_{n_1, n_2} \rightarrow G_{k^{n_1}} \times G_{k^{n_2}}$. Denote the kernel by U_{n_1, n_2} . Define the functor $i_{n_1, n_2}: \mathrm{Rep}(G_{k^{n_1}}) \times \mathrm{Rep}(G_{k^{n_2}}) \rightarrow \mathrm{Rep}(G_{k^n})$ taking representations σ_1 and σ_2 of $G_{k^{n_1}}$ and $G_{k^{n_2}}$ respectively to the induction to G_{k^n} of the pull-back under φ of $\sigma_1 \otimes \sigma_2$. The functor $r_{n_1, n_2}: \mathrm{Rep}(G_{k^n}) \rightarrow \mathrm{Rep}(G_{k^{n_1}}) \times \mathrm{Rep}(G_{k^{n_2}})$ is defined by restricting a representation ρ of G_{k^n} to P_{n_1, n_2} and then taking the invariants under U_{n_1, n_2} . By Frobenius reciprocity, these functors form an adjoint pair:

$$\mathrm{Hom}_{G_{k^n}}(\rho, i_{n_1, n_2}(\sigma_1, \sigma_2)) = \mathrm{Hom}_{G_{k^{n_1}} \times G_{k^{n_2}}}(r_{n_1, n_2}(\rho), \sigma_1 \otimes \sigma_2).$$

Following [Onn07], the functors i_{n_1, n_2} and r_{n_1, n_2} are called *geometric induction* and *geometric restriction* functors, respectively. Furthermore

Definition 3.1. An irreducible representation of G_{k^n} will be said to lie in the geometrically induced series if it is isomorphic to a subrepresentation of $i_{n_1, n_2}(\sigma_1, \sigma_2)$ for some decomposition $n = n_1 + n_2$ with n_1 and n_2 strictly positive, and some representations σ_1 and σ_2 of $G_{k^{n_1}}$ and $G_{k^{n_2}}$ respectively.

3.2. Infinitesimal induction and restriction functors. For two partitions λ and μ , say that $\lambda \leq \mu$ if there exists an embedding of \mathfrak{o}_λ in \mathfrak{o}_μ as an \mathfrak{o} -module. This is equivalent to the existence of a surjective \mathfrak{o} -module morphism $\mathfrak{o}_\mu \rightarrow \mathfrak{o}_\lambda$. If $\lambda \leq k^n$, then the pair (λ, k^n) has the *unique embedding* and *unique quotient* properties, i.e., all embeddings of \mathfrak{o}_λ in \mathfrak{o}_{k^n} and all surjections of \mathfrak{o}_{k^n} onto \mathfrak{o}_λ lie in the same G_{k^n} -orbit. As a consequence the functors that are defined below will, up to isomorphism, not depend on the choices of embeddings and surjections involved (in the language of [BO07, Section 2], k^n is a *symmetric type*).

Given $\lambda \leq k^n$, take the obvious embedding of \mathfrak{o}_λ in \mathfrak{o}_k^n given on standard basis vectors by $\mathbf{f}_i \mapsto \pi^{k-\lambda_h(i)} \mathbf{e}_i$, where $h(i)$ is such that $r_1 + \cdots + r_{h(i)-1} < i \leq r_1 + \cdots + r_{h(i)}$. Define

$$P_{\lambda \hookrightarrow k^n} = \{g \in G_{k^n} \mid g \cdot \mathfrak{o}_\lambda = \mathfrak{o}_\lambda\},$$

Restriction to \mathfrak{o}_λ gives rise to a homomorphism $P_{\lambda \hookrightarrow k^n} \rightarrow G_\lambda$ which, due to the unique embedding property, is surjective. Let $U_{\lambda \hookrightarrow k^n}$ be the kernel. One may now define an induction functor $i_{\lambda \hookrightarrow k^n}: \mathrm{Rep}(G_\lambda) \rightarrow \mathrm{Rep}(G_{k^n})$ as follows: given a representation of G_λ , pull it back to a representation of $P_{\lambda \hookrightarrow k^n}$ via the homomorphism $P_{\lambda \hookrightarrow k^n} \rightarrow G_\lambda$, and then induce to G_{k^n} . Its adjoint functor $r_{\lambda \hookrightarrow k^n}: \mathrm{Rep}(G_{k^n}) \rightarrow \mathrm{Rep}(G_\lambda)$ is obtained by taking a representation of G_{k^n} , restricting to $P_{\lambda \hookrightarrow k^n}$, and taking the vectors invariant under $U_{\lambda \hookrightarrow k^n}$. The adjointness is a version of Frobenius reciprocity: there is a natural isomorphism

$$\mathrm{Hom}_{G_{k^n}}(\rho, i_{\lambda \hookrightarrow k^n}(\sigma)) = \mathrm{Hom}_{G_\lambda}(r_{\lambda \hookrightarrow k^n}(\rho), \sigma)$$

for representations ρ and σ of G_{k^n} and G_λ respectively. In terms of matrices, the groups $P_{\lambda \hookrightarrow k^n}$ and $U_{\lambda \hookrightarrow k^n}$ are

$$P_{\lambda \hookrightarrow k^n} = \{(a_{ij}) \in G_{k^n} \mid a_{ij} \in \pi^{\min\{0, \lambda_h(j) - \lambda_h(i)\}} \mathfrak{o}_k\},$$

$$U_{\lambda \hookrightarrow k^n} = \{(a_{ij}) \in P_{\lambda \hookrightarrow k^n} \mid a_{ij} \in \delta_{ij} + \pi^{\lambda_h(j)} \mathfrak{o}_k\}.$$

Dually, fix the surjection of \mathfrak{o}_k^n onto \mathfrak{o}_λ given by $\mathbf{e}_i \mapsto \mathbf{f}_i$ and define

$$P_{k^n \twoheadrightarrow \lambda} = \{g \in G_{k^n} \mid g \cdot \ker(\mathfrak{o}_k^n \rightarrow \mathfrak{o}_\lambda) = \ker(\mathfrak{o}_k^n \rightarrow \mathfrak{o}_\lambda)\}.$$

Taking the induced map on the quotient gives rise to a homomorphism $P_{k^n \twoheadrightarrow \lambda} \rightarrow G_\lambda$ which, by the unique quotient property, is surjective. Let $U_{k^n \twoheadrightarrow \lambda}$ denote the kernel. An adjoint pair of functors $i_{k^n \twoheadrightarrow \lambda}: \text{Rep}(G_\lambda) \rightarrow \text{Rep}(G_{k^n})$ and $r_{k^n \twoheadrightarrow \lambda}: \text{Rep}(G_{k^n}) \rightarrow \text{Rep}(G_\lambda)$ are defined exactly as before. $P_{k^n \twoheadrightarrow \lambda}$ is conjugate to $P_{\lambda' \hookrightarrow k^n}$ and $U_{k^n \twoheadrightarrow \lambda}$ is conjugate to $U_{\lambda' \hookrightarrow k^n}$, where λ' is the partition that is complementary to λ in k^n , i.e., the partition for which $\ker(\mathfrak{o}_k^n \rightarrow \mathfrak{o}_\lambda) \cong \mathfrak{o}_{\lambda'}$. Therefore, the collection of irreducible representations obtained as summands after applying either of the functors $i_{\lambda \hookrightarrow k^n}$ or $i_{k^n \twoheadrightarrow \lambda}$ is the same. Following [Onn07], the functors $i_{\lambda \hookrightarrow k^n}$ and $i_{k^n \twoheadrightarrow \lambda}$ are called *infinitesimal induction functors*. The functors $r_{\lambda \hookrightarrow k^n}$ and $r_{k^n \twoheadrightarrow \lambda}$ are called *infinitesimal restriction functors*.

Definition 3.2. An irreducible representation of G_{k^n} will be said to lie in the infinitesimally induced series if it is isomorphic to a subrepresentation of $i_{\lambda \hookrightarrow k^n} \sigma$ for some partition $\lambda \leq k^n$ and some representation σ of G_λ .

4. CUSPIDALITY AND STRONG CUSPIDALITY

4.1. The definitions of cuspidality. Recall from Section 2.2 that to every irreducible representation ρ of G_{k^n} is associated a similarity class $\Omega_1(\rho) \subset M_n(\mathfrak{o}_1)$. The following definition was introduced in [Kut80] for $n = 2$ and in [Car84] for general n .

Definition 4.1 (Strong cuspidality). An irreducible representation ρ of G_{k^n} is said to be *strongly cuspidal* if either $k = 1$ and ρ is cuspidal, or $k > 1$ and $\Omega_1(\rho)$ is an irreducible orbit in $M_n(\mathfrak{o}_1)$.

In the above definition, one says that an orbit is irreducible if the matrices in it are irreducible, i.e., they do not leave any non-trivial proper subspaces of \mathfrak{o}_1^n invariant. This is equivalent to saying that the characteristic polynomial of any matrix in the orbit is irreducible.

Another notion of cuspidality (which applies for any G_λ , however, we shall focus on $\lambda = k^n$) picks out those irreducible representations which can not be constructed from the representations of G_λ , $\lambda \leq k^n$ by using the functors defined in Section 3.

Definition 4.2 (Cuspidality). An irreducible representation ρ of G_{k^n} is said to be *cuspidal* if no twist of it by a linear character lies in the geometrically or infinitesimally induced series.

4.2. Comparison between the definitions.

Theorem 4.3. *Every strongly cuspidal representation is cuspidal. When n is a prime, every cuspidal representation is strongly cuspidal.*

Proof. Let ρ be an irreducible non-cuspidal representation of G_{k^n} . The linear characters of G_{k^n} are of the form $\det \circ \chi$ for some character $\chi: \mathfrak{o}_k^\times \rightarrow \mathbf{C}^\times$. Using the identification of $N_{k-1} \simeq M_n(\mathfrak{o}_1)$ with its dual from Section 2.2, the restriction of $\det \circ \chi$ to N_{k-1} is easily seen to be a scalar matrix. Thus ρ is strongly cuspidal if and only if $\rho(\chi) = \rho \otimes \det \circ \chi$ is, since adding a scalar matrix does not effect the irreducibility of the orbit $\Omega_1(\rho)$. Since ρ is non-cuspidal, there exists a character χ such that $\rho(\chi)^U$ is nonzero for some $U = U_{n_1, n_2}$ or $U = U_{\lambda \rightarrow k^n}$. In either case this implies that the orbit $\Omega_1(\rho(\chi))$ is reducible which in turn implies that $\rho(\chi)$ and hence ρ are not strongly cuspidal.

For the converse the following interesting result (for which the hypothesis that n is prime is not necessary) plays an important role. A similar result was obtained by Kutzko in the context of supercuspidal representations of GL_n over a p -adic field [Kut85, Prop. 4.6]. Call a similarity class in $M_n(\mathfrak{o}_1)$ *primary* if its characteristic polynomial has a unique irreducible factor.

Proposition 4.4. *Let ρ be an irreducible representation of G_{k^n} . If $\Omega_1(\rho)$ is not primary then ρ lies in the geometrically induced series.*

Proof. If $\Omega_1(\rho)$ is not primary then it contains an element $\varphi = \begin{pmatrix} \hat{w}_1 & 0 \\ 0 & \hat{w}_2 \end{pmatrix}$ with $\hat{w}_i \in M_{n_i}(\mathfrak{o}_1)$ and $n = n_1 + n_2$, such that the characteristic polynomials of \hat{w}_1 and \hat{w}_2 have no common factor. It will be shown that $r_{n_1, n_2}(\rho) \neq 0$.

In what follows, matrices will be partitioned into blocks according to $n = n_1 + n_2$. Let $P_i = P_{(k^{n_1}, (k-i)n_2) \hookrightarrow k^n}$ for $i = 0, \dots, k$. Then P_i consists of matrices in G_{k^n} with blocks of the form $\begin{pmatrix} a & b \\ \varpi^i c & d \end{pmatrix}$. Let U_i be the normal subgroup of P_i consisting of block matrices of the form $\begin{pmatrix} 1 & \varpi^{k-i} u \\ 0 & 1 \end{pmatrix}$. The P_i 's form a decreasing sequence of subgroups, while the U_i 's form increasing sequences. Given a representation ρ_i of P_i/U_i define $r_i(\rho_i)$ to be the representation of P_{i+1}/U_{i+1} obtained by taking the vectors in the restriction of ρ_i to P_{i+1} that are invariant under U_{i+1} . That is,

$$r_i: \mathrm{Rep}(P_i/U_i) \rightarrow \mathrm{Rep}(P_{i+1}/U_{i+1}), \quad r_i(\rho_i) = \mathrm{Inv}_{U_{i+1}/U_i} \circ \mathrm{Res}_{P_{i+1}/U_i}^{P_i/U_i}(\rho_i).$$

In particular, $P_k = P_{n_1, n_2}$ and $U_k = U_{n_1, n_2}$. Therefore, (see [Onn07, Lemma 7.1]) we have that $r_{n_1, n_2} = r_{k-1} \circ \dots \circ r_0$. We argue by induction that $r_i \circ \dots \circ r_0(\rho) \neq 0$ for all $i = 0, \dots, k$. If $i = 0$, then since $\varphi \in \Omega_1(\rho)$, we get that $\rho|_{U_1}$ contains the trivial character of U_1 , hence, $r_0(\rho) \neq 0$. Denote $\rho_i = r_{i-1} \circ \dots \circ r_0(\rho)$ and assume that $\rho_i \neq 0$. In order to show that $r_i(\rho_i) \neq 0$, consider the normal subgroup L_i of P_i which consists of block matrices of the form $I + \begin{pmatrix} \varpi^{k-1} w_1 & \varpi^{k-i-1} u \\ \varpi^{k-1} v & \varpi^{k-1} w_2 \end{pmatrix}$. It is easily verified that $L_i/U_i \simeq M_n(\mathfrak{o}_1)$, the isomorphism given by

$$\eta: I + \begin{pmatrix} \varpi^{k-1} w_1 & \varpi^{k-i-1} u \\ \varpi^{k-1} v & \varpi^{k-1} w_2 \end{pmatrix} \pmod{U_i} \mapsto \begin{pmatrix} w_1 & u \\ v & w_2 \end{pmatrix},$$

where w_1, w_2, u and v are appropriate block matrices over \mathfrak{o}_1 . It follows that we can identify the dual of L_i/U_i with $M_n(\mathfrak{o}_1)$: $\hat{x} \mapsto \psi_{\hat{x}} \circ \eta$, for $\hat{x} \in M_n(\mathfrak{o}_1)$.

The action of P_i on the dual of L_i/U_i is given by $\hat{x} \mapsto g\hat{x}$ where $\psi_{g\hat{x}}(\eta(l)) = \psi_{\hat{x}}(\eta(g^{-1}lg))$. We shall not need the general action of elements of P_i , but rather of a small subgroup which is much easier to handle. If

$$g_c = \begin{pmatrix} I & \\ \varpi^i c & I \end{pmatrix}, \quad \eta(l) = \begin{pmatrix} w_1 & u \\ v & w_2 \end{pmatrix}, \quad \hat{x} = \begin{pmatrix} \hat{w}_1 & \hat{v} \\ \hat{u} & \hat{w}_2 \end{pmatrix},$$

then unraveling definitions gives

$$(4.5) \quad \hat{x} \mapsto g_c \hat{x} = \begin{pmatrix} \hat{w}_1 & \hat{v} \\ \hat{u} + c\hat{w}_1 - \hat{w}_2 c & \hat{w}_2 \end{pmatrix}.$$

As we have identifications $L_0/U_1 = \cdots = L_i/U_{i+1}$ we infer that the restriction of ρ_i to L_i/U_{i+1} contains a character

$$\psi_{\hat{x}} = (\varphi|_{L_0/U_1}, \hat{u}): L_i/U_{i+1} \times U_{i+1}/U_i = L_i/U_i \rightarrow \mathbf{C}^\times,$$

that is, $\psi_{\hat{x}}$ corresponds to $\hat{x} = \begin{pmatrix} \hat{w}_1 & 0 \\ \hat{u} & \hat{w}_2 \end{pmatrix}$. We claim that there exist g_c such that

$$g_c \hat{x} = \begin{pmatrix} \hat{w}_1 & 0 \\ 0 & \hat{w}_2 \end{pmatrix},$$

therefore $\rho_i|_{U_{i+1}/U_i}$ contains the trivial character of U_{i+1}/U_i and hence $r_i(\rho_i) \neq 0$.

Indeed, using (4.5) it is enough to show that the map $c \mapsto c\hat{w}_1 - \hat{w}_2 c$ is surjective, hence \hat{u} can be eliminated and the entry $(1, 2)$ contains the trivial character. This map is surjective if and only if it is injective. So we show that its kernel is null. A matrix c is in the kernel if and only if

$$(4.6) \quad c\hat{w}_1 = \hat{w}_2 c.$$

Let p_i ($i = 1, 2$) be the characteristic polynomials of \hat{w}_i . Our assumption on the orbits is that p_1 and p_2 have disjoint set of roots. Using (4.6) we deduce that

$$cp_1(\hat{w}_1) = p_1(\hat{w}_2)c.$$

By the Cayley-Hamilton theorem the left hand side of the above equation vanishes. Over an algebraic closure of \mathfrak{o}_1 , $p_1(t) = \prod (t - \alpha_j)$, where the α_j are the roots of p_1 . The hypothesis on \hat{w}_1 and \hat{w}_2 implies that none of these is an eigenvalue of \hat{w}_2 . Therefore, $\hat{w}_2 - \alpha_j$ is invertible for each j . It follows that $p_1(\hat{w}_2) = \prod (\hat{w}_2 - \alpha_j)$ is also invertible, hence $c = 0$. This completes the proof of the proposition. □

Returning now to the proof of Theorem 4.3, assume that ρ is not strongly cuspidal. There are two possibilities:

- (a) Any element $\hat{\omega} \in \Omega_1(\rho)$ has eigenvalue in \mathfrak{o}_1 . In such case, by twisting with a one-dimensional character χ , we get a row of zeros in the Jordan canonical form of $\hat{\omega}$. Therefore, $\rho(\chi)$ is contained in a representation infinitesimally induced from $G_{(k^{n-1}, k-1)}$.
- (b) Elements in $\Omega_1(\rho)$ have no eigenvalue in \mathfrak{o}_1 . Since n is prime and since $\Omega_1(\rho)$ is reducible, the latter cannot be primary, and Proposition 4.4 implies that ρ lies in the geometrically induced series.

Thus, ρ is non-cuspidal. □

5. CONSTRUCTION OF STRONGLY CUSPIDAL REPRESENTATIONS

The construction of strongly cuspidal representations of $\mathrm{GL}_n(\mathfrak{o}_k)$ when $k > 1$ can be found, for example, in [Shi68, Gér75, How77, Car84, BK93, Hil95a]. In this section, we recall this construction in a way that Theorems B and C are seen to follow from it.

5.1. Primitive characters. Let E denote an unramified extension of F of degree n . Let \mathfrak{O} be the integral closure of \mathfrak{o} in E . The maximal ideal of \mathfrak{O} is $\mathfrak{P} = \varpi\mathfrak{O}$. Let $\mathfrak{O}_k = \mathfrak{O}/\mathfrak{P}^k$. As an \mathfrak{o}_k -module, \mathfrak{O}_k is isomorphic to a free \mathfrak{o}_k -module of rank n . Therefore, G_{k^n} can be identified with $\text{Aut}_{\mathfrak{o}_k}(\mathfrak{O}_k)$. This identification is determined up to an inner automorphism of G_{k^n} . Thus, the strongly cuspidal representations constructed in this section are determined up to isomorphism.

Left multiplication by elements of \mathfrak{O}_k gives rise to \mathfrak{o}_k -module endomorphisms of \mathfrak{O}_k . Therefore, \mathfrak{O}_k^\times can be thought of as a subgroup of G_{k^n} . Similarly, for each $r \geq k/2$, \mathfrak{O}_{k-r} will be thought of as a subring of $M_n(\mathfrak{o}_{k-r})$.

Strongly cuspidal representations of G_{k^n} will be associated to certain characters of \mathfrak{O}_k^\times which we will call *strongly primitive*. In order to define a strongly primitive character of \mathfrak{O}_k^\times it is first necessary to define a primitive character of \mathfrak{O}_1 .

Definition 5.1 (Primitive character of \mathfrak{O}_1). A *primitive character* of \mathfrak{O}_1 is a homomorphism $\phi: \mathfrak{O}_1 \rightarrow \mathbf{C}^\times$ which does not factor through any proper subfield via the trace map.

The map $\mathfrak{O}_k \rightarrow \mathfrak{O}_k^\times$ given by $a \mapsto 1 + \varpi^r a$ induces an isomorphism $\mathfrak{O}_{k-r} \xrightarrow{\sim} \ker(\mathfrak{O}_k^\times \rightarrow \mathfrak{O}_r^\times)$, for each $1 \leq r < k$.

Definition 5.2 (Strongly primitive character of \mathfrak{O}_k^\times). When $k > 1$, a *strongly primitive character* of \mathfrak{O}_k^\times is a homomorphism $\omega: \mathfrak{O}_k^\times \rightarrow \mathbf{C}^\times$ whose restriction to $\ker(\mathfrak{O}_k^\times \rightarrow \mathfrak{O}_{k-1}^\times)$ is a primitive character when thought of as a character of \mathfrak{O}_1 under the above identification.

The above definition does not depend on the choice of uniformizing element $\varpi \in \mathfrak{p}$. Suppose that $r \geq k/2$. An identification $A \mapsto \psi_A$ of $M_n(\mathfrak{o}_{k-r})$ with its Pontryagin dual was constructed in Section 2.2. Given $a \in \mathfrak{O}_{k-r}$, view it as an element of $M_n(\mathfrak{o}_{k-r})$. Let ϕ_a denote the restriction of ψ_a to \mathfrak{O}_{k-r} . Then $a \mapsto \phi_a$ is an isomorphism of \mathfrak{O}_{k-r} with its Pontryagin dual.

5.2. Construction of strongly cuspidal representations from strongly primitive characters. The reader may find it helpful to refer to (5.8) while navigating the construction. Let $l = \lceil k/2 \rceil$ be the smallest integer not less than $k/2$ and $l' = \lfloor k/2 \rfloor$ be the largest integer not greater than $k/2$. Let ω be a strongly primitive character of \mathfrak{O}_k^\times . Let $a \in \mathfrak{O}_{k-l}$ be such that the restriction of ω to $N_l \cap \mathfrak{O}_k^\times$ (when identified with \mathfrak{O}_{k-l}) is of the form ϕ_a . The strong primitivity of ω implies that the image of a in \mathfrak{O}_1 does not lie in any proper subfield. The formula

$$(5.3) \quad \tau_\omega(xu) = \psi_a(x)\omega(u) \text{ for all } x \in N_l \text{ and } u \in \mathfrak{O}_k^\times,$$

defines a homomorphism $\tau_\omega: N_l \mathfrak{O}_k^\times \rightarrow \mathbf{C}^\times$. Let L denote the kernel of the natural map $\mathfrak{O}_k^\times \rightarrow \mathfrak{O}_1^\times$. Then $N_l L$ is a normal subgroup of $N_{l'} \mathfrak{O}_k^\times$ (note that $N_l \mathfrak{O}_k^\times$ is not normal in $N_{l'} \mathfrak{O}_k^\times$, when k is odd). Let σ_ω denote the restriction of τ_ω to $N_l L$. We have

$$(5.4) \quad \sigma_\omega(yxy^{-1}) = \sigma_\omega(x) \text{ for all } y \in N_{l'} \mathfrak{O}_k^\times \text{ and } x \in N_l L.$$

Let q denote the order and p denote the characteristic of \mathfrak{o}_1 . The quotient $V = N_{l'} L / N_l L$ is naturally isomorphic to $M_n(\mathfrak{o}_1) / \mathfrak{O}_1$ which¹, being an abelian group where every non-trivial element has order p , can be viewed as a vector space over \mathbf{F}_p of dimension $(n^2 - n) \log_p q$. Then

$$\beta(xN_l L, yN_l L) = \sigma_\omega([x, y]) \text{ for all } x, y \in N_{l'} L,$$

¹Here $M_n(\mathfrak{o}_1)$ is identified with $\text{End}_{\mathfrak{o}_1}(\mathfrak{O}_1)$.

defines a non-degenerate alternating bilinear form $\beta: V \times V \rightarrow \mu_p$, where μ_p denote the complex p^{th} roots of unity, [Hil95b, Corollary 4.3].

The following lemma now follows from standard results on the representation theory of finite Heisenberg groups (see e.g., [Gér75, Proposition 3]).

Lemma 5.5. *There exists a unique irreducible representation σ'_ω of $N_{l'}L$ whose restriction to N_lL is σ_ω isotypic. This representation has dimension $q^{(l-l')(n^2-n)/2}$. Its character is given by*

$$\text{tr}(\sigma'_\omega(x)) = \begin{cases} q^{(l-l')(n^2-n)/2} \sigma_\omega(x) & \text{if } x \in N_lL, \\ 0 & \text{otherwise.} \end{cases}$$

Recall from [Ser68, II.4, Proposition 8], that there is a unique multiplicative section $s: \mathfrak{D}_1^\times \rightarrow \mathfrak{D}_k^\times$. This allows us to realize $N_{l'}\mathfrak{D}_k^\times$ as a semidirect product of $N_{l'}L$ by \mathfrak{D}_1^\times . Recall also, that $x \in \mathfrak{D}_k^\times$ is called *regular elliptic* if its image in \mathfrak{D}_1^\times is not contained in any proper subfield.

Lemma 5.6. *When k is odd, there exists an irreducible representation τ'_ω of $N_{l'}\mathfrak{D}_k^\times$, which is unique up to isomorphism, whose restriction to N_lL is σ_ω -isotypic, and such that for any $x \in N_{l'}\mathfrak{D}_k^\times$,*

$$\text{tr}(\tau'_\omega(x)) = \begin{cases} 0 & \text{when } x \text{ is not conjugate to an element of } N_l\mathfrak{D}_k^\times \\ (-1)^{n-1} \omega(x) & \text{when } x \in \mathfrak{D}_k^\times \text{ is regular elliptic.} \end{cases}$$

Proof. The lemma is easily deduced from [Gér75, Theorem 1] as follows: the algebraic torus T defined over \mathbf{F}_q such that $T(\mathbf{F}_q) = \mathfrak{D}_1^\times$ splits over the extension \mathbf{F}_{q^n} of \mathbf{F}_q . The Galois group of this extension acts on the weights of $T(\mathbf{F}_{q^n})$ on $V \otimes \mathbf{F}_{q^n}$, which simply correspond to roots of GL_n . The Frobenius automorphism which generates this group acts as a Coxeter element on this root system. One may see that, in the language of [Gér75, 1.4.9(b)], this action has a unique symmetric orbit and $(n-2)/2$ non-symmetric orbits if n is even, and no symmetric orbits and $(n-1)/2$ non-symmetric orbits if n is odd. The symmetric orbits contribute a factor of (-1) to the character values. The hypothesis that u is not an element of any proper subfield of \mathfrak{D}_1 implies that u is regular semisimple, and that no weight vanishes on it. \square

When k is even, define the representation τ'_ω of $N_{l'}\mathfrak{D}_k^\times$ to be just τ_ω (see (5.3)). Then, for any $k > 1$, if $u \in \mathfrak{D}_k^\times$ is an element whose image in \mathfrak{D}_1^\times generates \mathfrak{D}_1^\times , we have

$$(5.7) \quad \text{tr}(\tau'_\omega(u)) = (-1)^{k(n-1)} \omega(u).$$

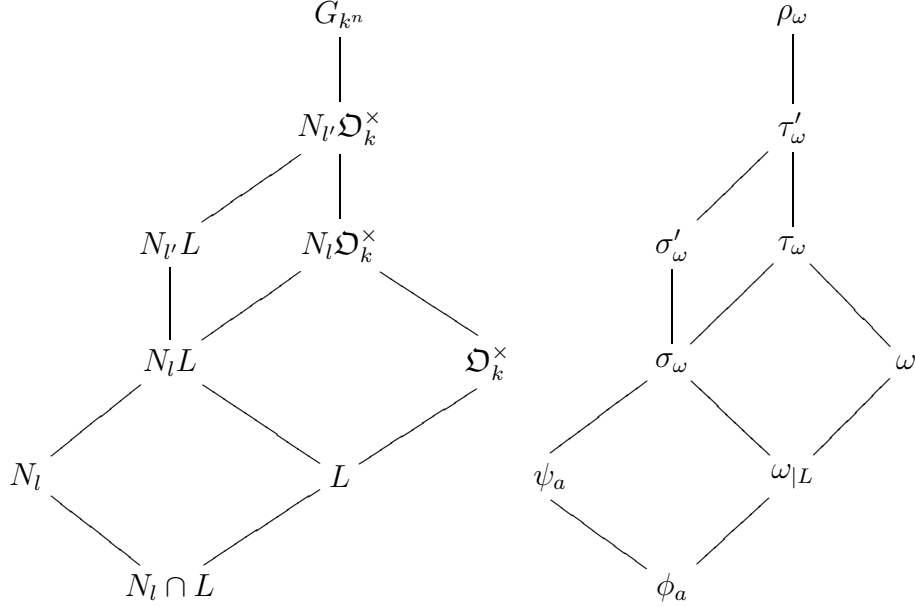
Finally, define

$$\rho_\omega = \text{Ind}_{N_{l'}\mathfrak{D}_k^\times}^{G_{k^n}} \tau'_\omega.$$

This will be the strongly cuspidal representation associated to the strongly primitive character ω of \mathfrak{D}_k^\times . The representation ρ_ω is irreducible because $N_{l'}\mathfrak{D}_k^\times$ is the centralizer of σ_ω in G_{k^n} .

The steps in the construction of ρ_ω are described schematically below for the convenience of the reader. The diagram on the left describes the relation between the various groups involved. The position occupied by a group in the diagram on the left is occupied by the corresponding representation that appears in the construction in the diagram on the right.

(5.8)



Theorem 5.9. *For each strongly primitive character ω of \mathfrak{D}_k^\times , ρ_ω is an irreducible representation such that*

- (1) $\text{tr}(\rho_\omega(g)) = 0$ if g is not conjugate to an element of $N_l\mathfrak{D}_k^\times$.
- (2) if $u \in \mathfrak{D}_k^\times$ is such that its image in \mathfrak{D}_1^\times is not contained in any proper subfield, then

$$\text{tr}(\rho_\omega(u)) = (-1)^{k(n-1)} \sum_{\gamma \in \text{Gal}(E/F)} \omega(\gamma u).$$

for every $u \in \mathfrak{D}_k^\times$, whose image in \mathfrak{D}_1^\times lies in no proper subfield.

Proof. The first assertion follows from Lemma 5.5. The second follows from the fact that the intersection of the conjugacy class of u in G_{k^n} with \mathfrak{D}_k^\times consists only of the elements γu , for $\gamma \in \text{Gal}(E/F)$. \square

5.3. The parameterization of strongly cuspidal representations of G_{k^n} . The following is a detailed version of Theorem B.

Theorem 5.10.

- (1) *For each strongly primitive character ω of \mathfrak{D}_k^\times , the representation ρ_ω of G_{k^n} is irreducible and strongly cuspidal.*
- (2) *Every strongly cuspidal representation of G_{k^n} is isomorphic to ρ_ω for some strongly primitive character ω of \mathfrak{D}_k^\times .*
- (3) *If ω' is another strongly primitive character of \mathfrak{D}_k^\times , then ρ_ω is isomorphic to $\rho_{\omega'}$ if and only if $\omega' = \omega \circ \gamma$ for some $\gamma \in \text{Gal}(E/F)$.*

Proof of (1). The irreducibility of ρ_ω follows from standard results on induced representations. To see that ρ_ω is strongly cuspidal, observe that the restriction of ρ_ω to N_l contains ψ_a . This means that its restriction to N_{k-1} contains $\psi_{\bar{a}}$, where \bar{a} is the image of a in \mathfrak{D}_1 . Since this image does not lie in any proper subfield, its minimal polynomial is irreducible of degree n . Therefore, as an element of $M_n(\mathfrak{o}_1)$, its characteristic polynomial must be irreducible. \square

Proof of (2). Suppose that ρ is an irreducible strongly cuspidal representation of G_{k^n} . Unwinding the definitions, one sees that $\Omega_1(\rho)$ is just the image of $\Omega_{k-l}(\rho)$ under the natural map $M_n(\mathfrak{o}_{k-l}) \rightarrow M_n(\mathfrak{o}_1)$. Let $p(t) \in \mathfrak{o}_{k-l}[t]$ be the characteristic polynomial of the matrices in $\Omega_{k-l}(\rho)$. Denote its image in $\mathfrak{o}_1[t]$ by $\bar{p}(t)$. The hypothesis on ρ implies that $\bar{p}(t)$ is irreducible. Let $\tilde{p}(t)$ be any polynomial in $\mathfrak{o}[t]$ whose image in $\mathfrak{o}_{k-l}[t]$ is $p(t)$. By Hensel's lemma, there is a bijection between the roots of $\tilde{p}(t)$ in E and the roots of $\bar{p}(t)$ in \mathfrak{D}_1 . Consequently,

$$\mathrm{Hom}_F(F[t]/\tilde{p}(t), E) \cong \mathrm{Hom}_{\mathfrak{o}_1}(\mathfrak{o}_1[t]/\bar{p}(t), \mathfrak{D}_1).$$

But we know that \mathfrak{D}_1 is isomorphic to $\mathfrak{o}_1[t]/\bar{p}(t)$. In fact there are exactly n such isomorphisms. Each one of these gives an embedding of $F[t]/\tilde{p}(t)$ in E . Since both $F[t]/\tilde{p}(t)$ and E have degree n , these embeddings must be isomorphisms. Any root \tilde{a} of $\tilde{p}(t)$ in E also lies in \mathfrak{D} . It is conjugate to the companion matrix of $\tilde{p}(t)$ in $\mathrm{GL}_n(\mathfrak{o})$. Therefore, its image $a \in \mathfrak{D}_{k-l}$ lies in $\Omega_{k-l}(\rho)$. It follows that $\rho|_{N_l}$ contains a ψ_a isotypic vector.

By applying the little groups method of Wigner and Mackey to the normal subgroup N_l of G_{k^n} , we see that every representation of ρ_k whose restriction to N_l has a ψ_a isotypic vector is induced from an irreducible representation of $N_l \mathfrak{D}_k^\times$ whose restriction to N_l is ψ_a isotypic. It is not difficult then to verify (by counting extensions at each stage) that the construction of τ'_ω in Section 5.2 gives all such representations. \square

Proof of (3). It follows from the proof of (2) that τ'_{ω_1} and τ'_{ω_2} are isomorphic if and only if $\omega_1 = \omega_2$. The Galois group $\mathrm{Gal}(E/F)$ acts by inner automorphisms of G_{k^n} (since we have identified it with $\mathrm{Aut}_{\mathfrak{o}_k}(\mathfrak{D}_k)$) preserving $N_l \mathfrak{D}_k^\times$. Therefore, the restriction of ρ_{ω_1} to $N_l \mathfrak{D}_k^\times$ also contains τ_{ω_2} whenever ω_2 is in the $\mathrm{Gal}(E/F)$ -orbit of ω_1 , hence ρ_{ω_1} is isomorphic to ρ_{ω_2} . If ω_1 and ω_2 do not lie in the same $\mathrm{Gal}(E/F)$ -orbit then Theorem 5.9 implies that ρ_{ω_1} can not be isomorphic to ρ_{ω_2} . \square

5.4. Connection with supercuspidal representations of $\mathrm{GL}_n(F)$. In [BK93, Theorem 8.4.1], Bushnell and Kutzko proved that all the irreducible supercuspidal representations of $\mathrm{GL}_n(F)$ can be obtained by compact induction from a compact subgroup modulo the center. One such subgroup is $F^\times \mathrm{GL}_n(\mathfrak{o})$. This group is a product of $\mathrm{GL}_n(\mathfrak{o})$ with the infinite cyclic group Z_1 generated by ϖI . Thus every irreducible representation of this group is a product of a character of Z_1 with an irreducible representation of $\mathrm{GL}_n(\mathfrak{o})$. An irreducible representation of $\mathrm{GL}_n(\mathfrak{o})$ is said to be of level $k-1$ if it factors through $\mathrm{GL}_n(\mathfrak{o}_k)$, but not through $\mathrm{GL}_n(\mathfrak{o}_{k-1})$. When n is prime, the representations of $\mathrm{GL}_n(\mathfrak{o})$ which give rise to supercuspidal representations are precisely those which are of level $k-1$, for some $k > 1$, and, when viewed as representations of $\mathrm{GL}_n(\mathfrak{o}_k)$, are strongly cuspidal. For $k=1$, they are just the cuspidal representations of $\mathrm{GL}_n(\mathfrak{o}_1)$. The corresponding representations of $Z \mathrm{GL}_n(\mathfrak{o})$ are called *très cuspidale de type k* by Carayol in [Car84, Section 4.1]. The construction that Carayol gives for these representations is the same as the one given here, except that the construction here is made canonical by using Gérardin's results.

Let χ be any character of Z_1 . Set

$$\pi_{\omega, \chi} := \mathrm{c}\text{-Ind}_{\mathrm{GL}_n(\mathfrak{o}) F^\times}^{\mathrm{GL}_n(F)} (\rho_\omega \otimes \chi).$$

These are the supercuspidal representations of $\mathrm{GL}_n(F)$ associated to ρ_ω .

Let $r: \mathrm{GL}_n(\mathfrak{o}) \rightarrow \mathrm{GL}_n(\mathfrak{o}_k)$ denote the homomorphism obtained by reduction modulo \mathfrak{p}^k . In the notation of [BK93], we have $r^{-1}(N_l L) = H^1(\beta, \mathfrak{A})$, $r^{-1}(N_{l'} L) = J^1(\beta, \mathfrak{A})$ and $r^{-1}(N_{l'} \mathfrak{D}_k^\times) = J(\beta, \mathfrak{A})$, where $\mathfrak{A} = M_n(\mathfrak{o})$ and $\beta \in M_n(F)$ is minimal (see [BK93, (1.4.14)]).

These groups are very special cases of the groups defined in [BK93, (3.1.14)]. The inflation η of $\sigma_{\omega'}$ to $J^1(\beta, \mathfrak{A})$ is a special case of the Heisenberg representation defined in [BK93, Prop. 5.1.1].

We will say that a supercuspidal representation π of $\mathrm{GL}_n(F)$ belongs to the *unramified series* if the field extension $F[\beta]$ of F is unramified (by [BK93, (1.2.4), (6.2.3) (i)], this is equivalent to saying that the \mathfrak{o} -order \mathfrak{A} occurring in the construction of π is maximal). When n is a prime number, Carayol has proved (see [Car84, Theorem 8.1 (i)]) that the representations $\pi_{\omega, \chi}$ give all the supercuspidal representations of $\mathrm{GL}_n(F)$ which belong to the unramified series. However, when n is composite, the strongly cuspidal representations are not sufficient in order to build all the supercuspidal representations in the unramified series of $\mathrm{GL}_n(F)$ (see for instance Howe's construction in [How77]). Since all the supercuspidal representations of $\mathrm{GL}_n(F)$ are known [BK93], it would be natural to try restricting them to $\mathrm{GL}_n(\mathfrak{o})$ and see if one gets cuspidal representations among the components. On the other hand we observe that our notion of cuspidality is in a sense stronger than the usual notion of supercuspidality for representations of $\mathrm{GL}_n(F)$, since supercuspidality can only see geometric induction.

6. COMPLEXITY OF THE CLASSIFICATION PROBLEM

In this section it will be shown that the representation theory of the family of groups G_{k^n} actually involves the much larger family, $G_{\lambda, E}$ ($\lambda \in \Lambda$, E/F unramified), which was defined in Section 2.1, even when $k = 2$.

Theorem 6.1. *Let $F = \mathbf{F}_q((\varpi))$ be a local function field. Then the problems of constructing all the irreducible representations of the following groups are equivalent:*

- (1) $G_{2^n, F}$ for all $n \in \mathbf{N}$.
- (2) $G_{k^n, F}$ for all $k, n \in \mathbf{N}$.
- (3) $G_{\lambda, E}$ for all partitions λ and all unramified extensions E of F .

Proof. Obviously (3) implies (2), which implies (1). That (1) implies (3) follows from the somewhat more precise formulation in Theorem 6.2. \square

Theorem 6.2. *Let F be a local function field. Then the problem of constructing all the irreducible representations of $G_{2^n, F}$ is equivalent to the problem of constructing all the irreducible representations of all the groups $G_{\lambda, E}$, where E ranges over all unramified extensions of F of degree d and λ ranges over all partitions such that $d(\lambda_1 r_1 + \cdots + \lambda_l r_l) \leq n$.*

Proof. When F is a local function field, G_{2^n} is isomorphic to the semidirect product of $\mathrm{GL}_n(\mathfrak{o}_1)$ by $M_n(\mathfrak{o}_1)$. The *little groups method* of Wigner and Mackey (see e.g., [Ser77, Prop. 25]) shows that constructing the irreducible representations of G_{2^n} is equivalent to constructing the irreducible representations of the centralizers in $\mathrm{GL}_n(\mathfrak{o}_1)$ of all the multiplicative characters of $M_n(\mathfrak{o}_1)$. Pick any χ for which the space V_χ of χ -isotypic vectors is non-zero. By the discussion in Section 2.2, these subgroups of $\mathrm{GL}_n(\mathfrak{o}_1)$ are the same as the centralizer groups of matrices. We will see below that these centralizer groups are products of groups of the form $G_{\lambda, E}$ that appear in the statement of Theorem 6.2.

Let $A \in M_n(\mathfrak{o}_1)$. Then, \mathfrak{o}_1^n can be thought of as a $\mathfrak{o}_1[\varpi]$ -module where ϖ acts through A . The centralizer of A is the automorphism group of this $\mathfrak{o}_1[\varpi]$ -module. For each irreducible

monic polynomial $f(\varpi) \in \mathfrak{o}_1[\varpi]$ of degree d which divides the characteristic polynomial of A , the f -primary part of this module is isomorphic to

$$(\mathfrak{o}_1[\varpi]/f(\varpi)^{\lambda_1})^{r_1} \oplus \cdots \oplus (\mathfrak{o}_1[\varpi]/f(\varpi)^{\lambda_l})^{r_l},$$

for some partition λ .

Lemma 6.3. *Let $\mathfrak{D}_1 = \mathfrak{o}_1[\varpi]/f(\varpi)$. The rings $\mathfrak{o}_1[\varpi]/f(\varpi)^k$ and $\mathfrak{D}_1[u]/u^k$ are isomorphic for every $k > 0$.*

Proof. It will be shown by induction that there exists a sequence $\{q_k(\varpi)\}$, in $\mathfrak{o}_1[\varpi]$ such that

- (1) $q_1(\varpi) = \varpi$,
- (2) $q_{k+1}(\varpi) \equiv q_k(\varpi) \pmod{f(\varpi)^k}$ for all $k > 0$, and,
- (3) $f(q_k(\varpi)) \in f(\varpi)^k$ for all $k > 0$.

For $k = 1$ the result is obvious. Suppose that $q_k(\varpi)$ has been constructed. Since \mathfrak{o}_1 is a perfect field and $f(\varpi)$ is irreducible, $f'(\varpi)$ is not identically 0. It follows that $f'(\varpi)$ does not divide $f(\varpi)$. Since $q_k(\varpi) \equiv \varpi \pmod{f(\varpi)}$, $f'(q_k(\varpi))$ does not divide $f(\varpi)$. Therefore, the congruence

$$f(q_k(\varpi)) + f(\varpi)^k h(\varpi) f'(q_k(\varpi)) \equiv 0 \pmod{f(\varpi)^{k+1}}$$

can be solved for $h(\varpi)$. Let $h_0(\varpi)$ be a solution. Take $q_{k+1}(\varpi) = q_k(\varpi) + f(\varpi)^k h_0(\varpi)$. The sequence $\{q_k(\varpi)\}$ constructed in this manner has the required properties.

Now note that $\mathfrak{D}_1[u]/u^k \cong \mathfrak{o}_1[\varpi, u]/(f(\varpi), u^k)$. One may define a ring homomorphism

$$\mathfrak{o}_1[\varpi, u]/(f(\varpi), u^k) \rightarrow \mathfrak{o}_1[\varpi]/f(\varpi)^k$$

by $\varpi \mapsto q_k(\varpi)$ and $u \mapsto f(\varpi)$. Since $q_k(\varpi) \equiv \varpi \pmod{f(\varpi)}$, ϖ lies in the image of this map, so it is surjective. As vector spaces over \mathfrak{o}_1 both rings have dimension kd . Therefore, it is an isomorphism. \square

It follows from Lemma 6.3 that the automorphism group of the f -primary part of \mathfrak{o}_1^n is $G_{\lambda, E}$, where E is an unramified extension of F of degree d . The automorphism group of the $\mathfrak{o}_1[\varpi]$ -module \mathfrak{o}_1^n is the product of the automorphism groups of its f -primary parts. Therefore, the centralizer of A in G_1^n is a product of groups of the form $G_{\lambda, E}$. Considerations of dimension show that $d(\lambda_1 r_1 + \cdots + \lambda_l r_l) \leq n$ for each $G_{\lambda, E}$ that occurs.

Conversely given λ and d satisfying the above inequality, take an irreducible polynomial $f(\varpi) \in \mathfrak{o}_1[\varpi]$ of degree d . Define

$$J_k(f) = \begin{pmatrix} C_f & 0 & 0 & \cdots & 0 & 0 \\ I_d & C_f & 0 & \cdots & 0 & 0 \\ 0 & I_d & C_f & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & C_f & 0 \\ 0 & 0 & 0 & \cdots & I_d & C_f \end{pmatrix}_{kd \times kd},$$

where C_f is any matrix with characteristic polynomial f . Let

$$A = J_{\lambda_1}(f)^{\oplus r_1} \oplus \cdots \oplus J_{\lambda_l}(f)^{\oplus r_l} \oplus J_k(\varpi - a),$$

where $a \in \mathfrak{o}_1$ is chosen so that $\varpi - a \neq f(\varpi)$ and $k = n - d(\lambda_1 r_1 + \cdots + \lambda_l r_l)$. The centralizer of A contains $G_{\lambda, E}$ as a factor. \square

7. THE ZETA FUNCTION ASSOCIATED TO G_{k^n}

In [Spr75], Springer attaches a zeta function to irreducible representations of $\mathrm{GL}_n(\mathfrak{o}_1)$, and proves that for cuspidal representations it satisfies a functional equation. Later on, Macdonald [Mac80] shows that a functional equation holds for any irreducible representation, provided that it has no 1-component, namely, it is not contained in $i_{n-1,1}(\rho, 1)$ for any representation ρ of $\mathrm{GL}_{n-1}(\mathfrak{o}_1)$. Moreover, Macdonald establishes a bijection between irreducible representations of $\mathrm{GL}_n(\mathfrak{o}_1)$, and equivalence classes of tamely ramified representations of the Weil-Deligne group W'_F , which preserves certain L and ε factors.

In this section we attach a zeta function to any irreducible representation of G_{k^n} and show that it satisfies a functional equation, provided that ρ does not lie in the infinitesimally induced series. We follow closely [Mac80] and make the necessary adaptations.

The map $F \rightarrow \mathbf{C}^\times$ given by $x \mapsto \psi(\pi^k x)$, when restricted to \mathfrak{o} , factors through an additive character ψ_k of \mathfrak{o}_k , which does not factor through \mathfrak{o}_{k-1} . Denote $G = G_{k^n}$ and $M = M_{k^n} = M_n(\mathfrak{o}_k)$, and let $\mathbf{C}(M)$ denote complex valued functions on M . For $f \in \mathbf{C}(M)$ define its Fourier transform by

$$\hat{f}(x) = |M|^{-1/2} \sum_{y \in M} f(y) \psi_k(\mathrm{tr}(xy)),$$

so that $\hat{\hat{f}}(x) = f(-x)$. Let (ρ, V) be a finite dimensional representation of G . For each $f \in \mathbf{C}(M)$ define the zeta-function

$$\mathcal{Z}(f, \rho) = \sum_{g \in G} f(g) \rho(g) \in \mathrm{End}_{\mathbf{C}}(V).$$

Also, for $x \in M$ let

$$\mathcal{W}(\rho, \psi; x) = |M|^{-1/2} \sum_{g \in G} \psi_k(\mathrm{tr}(gx)) \rho(g).$$

The following lemma is straightforward.

Lemma 7.1.

- (a) $\mathcal{Z}(f, \rho) = \sum_{x \in M} \hat{f}(-x) \mathcal{W}(\rho, \psi; x)$.
- (b) $\mathcal{W}(\rho, \psi; xg) = \rho(g)^{-1} \mathcal{W}(\rho, \psi; x)$.
- (c) $\mathcal{W}(\rho, \psi; gx) = \mathcal{W}(\rho, \psi; x) \rho(g)^{-1}$.

In particular, setting $x = 1$ in parts (b)-(c) of Lemma 7.1 shows that $\mathcal{W}(\rho, \psi; 1)$ commutes with $\rho(g)$ for all $g \in G$. Therefore, if ρ is irreducible, then $\mathcal{W}(\rho, \psi; 1)$ is a scalar multiple of $\rho(1)$. Following [Mac80] we write $\varepsilon(\rho, \psi) \rho(1) = \mathcal{W}(\check{\rho}, \psi; 1)$, where $\check{\rho}$ is the contragredient of ρ , i.e. $\check{\rho}(g) = {}^t \rho(g^{-1})$ and $\varepsilon(\rho, \psi)$ is a complex number.

Proposition 7.2. *Let ρ be an irreducible representation of G which does not lie in the infinitesimally induced series. Then $\mathcal{W}(\rho, \psi; x) = 0$ for all $x \in M \setminus G$.*

Proof. Let $H_x = \{g \in G \mid gx = x\}$. For $g \in H_x$ we have

$$\mathcal{W}(\rho, \psi; x) = \mathcal{W}(\rho, \psi; gx) = \mathcal{W}(\rho, \psi; x) \rho(g^{-1}) = \mathcal{W}(\rho, \psi; x) \rho(e_{H_x}),$$

where $\rho(e_{H_x}) = |H_x|^{-1} \sum_{g \in H_x} \rho(g)$. Hence, it suffices to show that $\rho(e_{H_x}) = 0$ for $x \in M \setminus G$. Since $\rho(e_{H_x})$ is the idempotent projecting V onto V^{H_x} , it is enough to show that the latter subspace is null. Let $\mu = (\mu_1, \dots, \mu_n)$ be the divisor type of x . Namely,

$0 \leq \mu_1 \leq \dots \leq \mu_n \leq k$, such that acting with G on the right and on the left gives: $gxh = d_\mu = \text{diag}(\varpi^{\mu_1}, \dots, \varpi^{\mu_n})$. Then $H_x = gH_{d_\mu}g^{-1}$. Now for any μ we have $H_{d_\mu} \supset H_{d_\nu}$, where $\nu = (0, 0, \dots, 0, 1)$. Therefore, it is enough to show that $V^{H_{d_\nu}} = (0)$. The subgroup H_{d_ν} is given explicitly by

$$H_{d_\nu} = \begin{bmatrix} I_{n-1} & \varpi^{k-1} \star \\ 0 & 1 + \varpi^{k-1} \star \end{bmatrix} = U_{(k^{n-1}, k-1) \hookrightarrow k^n} \text{ (see Section 3.2).}$$

It follows that $V^{H_{d_\nu}} = (0)$ if ρ does not lie in the infinitesimally induced series. \square

Theorem 7.3. *For all $f \in \mathbf{C}(M)$ and all irreducible representations ρ of G that do not lie in the infinitesimally induced series, we have*

$${}^t\mathcal{Z}(\hat{f}, \check{\rho}) = \varepsilon(\rho, \psi)\mathcal{Z}(f, \rho).$$

Proof. If ρ does not lie in the infinitesimally induced series then nor does $\check{\rho}$, and hence $\mathcal{W}(\check{\rho}, \psi; x) = 0$ for all $x \in M \setminus G$. We get

$$\begin{aligned} {}^t\mathcal{Z}(\hat{f}, \check{\rho}) &= \sum_{g \in G} \hat{f}(-g) {}^t\mathcal{W}(\check{\rho}, \psi; g) && \text{(by Lemma 7.1(a))} \\ &= \mathcal{W}(\check{\rho}, \psi; 1) \sum_{g \in G} f(g) \rho(g) = \varepsilon(\rho, \psi)\mathcal{Z}(f, \rho) && \text{(by Lemma 7.1(c)).} \end{aligned}$$

\square

The possibility of relating representations of G_{k^n} with some equivalence classes of representations of the Weil-Deligne group W'_F , and consequently extending Macdonald correspondence to higher level, seems very appealing. However, such correspondence, if exists, is expected to be much more involved in view of the complexity of the representation theory of G_{k^n} .

8. CUSPIDAL REPRESENTATIONS WHICH ARE NOT STRONGLY CUSPIDAL

In this section we give a description of all the cuspidal representations of $G_{2^4} \cong \text{GL}_4(\mathfrak{o}_2)$ in the sense of Definition 4.2. This shows in particular the existence of representations which are cuspidal, yet are not strongly cuspidal.

Let $\lambda = (2^4)$ and put $G = G_\lambda$. If π is a cuspidal representation of G , then by Proposition 4.4 it is primary, that is, its orbit in $M_4(\mathbf{F}_q)$ consists of matrices whose characteristic polynomial is of the form $f(X)^a$, where $f(X)$ is an irreducible polynomial. If $a = 1$, then π is strongly cuspidal (by definition), and such representations were described in Section 5. On the other hand, $f(X)$ cannot have degree 1, because then it would be infinitesimally induced from $G_{(2^3, 1)}$, up to 1-dimensional twist (cf. the end of the proof of Theorem 4.3). We are thus reduced to considering representations whose characteristic polynomial is a reducible power of a non-linear irreducible polynomial. In the situation we are considering, there is only one such possibility, namely the case where $f(X)$ is quadratic, and $a = 2$. Let η denote an element which generates the extension $\mathbf{F}_{q^2}/\mathbf{F}_q$. We consider $M_2(\mathbf{F}_{q^2})$ as embedded in $M_4(\mathbf{F}_q)$ via the embedding $\mathbf{F}_{q^2} \hookrightarrow M_2(\mathbf{F}_q)$, by choosing the basis $\{1, \eta\}$ for \mathbf{F}_{q^2} over \mathbf{F}_q . Rational canonical form implies that in $M_4(\mathbf{F}_q)$ there are two conjugation orbits containing elements with two equal irreducible 2×2 blocks on the diagonal, one regular,

and one which is not regular (we shall call the latter *irregular*), represented by the following elements, respectively:

$$\beta_1 = \begin{pmatrix} \eta & 1 \\ 0 & \eta \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix}.$$

Therefore, any irreducible cuspidal non-strongly cuspidal representation of G has exactly one of the elements β_1 or β_2 in its orbit.

Denote by $N_1 \cong 1 + \varpi M_2(\mathfrak{o}_2)$ the kernel of the reduction map $G = G_{2^4} \rightarrow G_{1^4}$. As in Section 2.2, let ψ be a fixed non-trivial additive character of F , trivial on \mathfrak{o} . Then for each $\beta \in M_4(\mathbf{F}_q)$ we have a character $\psi_\beta : N_1 \rightarrow \mathbf{C}^\times$ defined by

$$\psi_\beta(1 + \varpi x) = \psi(\varpi^{-1} \text{Tr}(\beta x)).$$

The group G acts on its normal subgroup N_1 via conjugation, and thus on the set of characters of N_1 via the “coadjoint action”. For any character ψ_β of N_1 , we write

$$G(\psi_\beta) := \text{Stab}_G(\psi_\beta).$$

By Proposition 2.3 in [Hil95b], the stabilizer $G(\psi_\beta)$ is the preimage in G of the centralizer $C_{G_{1^4}}(\beta)$, under the reduction mod \mathfrak{p} map.

By definition, an irreducible representation π of G is cuspidal iff none of its 1-dimensional twists $\pi \otimes \chi \circ \det$ has any non-zero vectors fixed under any group $U_{i,j}$ or $U_{\lambda \hookrightarrow 2^4}$, or equivalently (by Frobenius reciprocity), if $\pi \otimes \chi \circ \det$ does not contain the trivial representation $\mathbf{1}$ when restricted to $U_{i,j}$ or $U_{\lambda \hookrightarrow 2^4}$. The groups $U_{i,j}$ are analogs of unipotent radicals of (proper) maximal parabolic subgroups of G , and $U_{\lambda \hookrightarrow 2^4}$ are the infinitesimal analogs of unipotent radicals (cf. Section 3). Note that since $\text{Ind}_{U_{i,j}}^G \mathbf{1} = \text{Ind}_{U_{i,j}}^G (\mathbf{1} \otimes \chi \circ \det) = (\text{Ind}_{U_{i,j}}^G \mathbf{1}) \otimes \chi \circ \det$, for any character $\chi : \mathfrak{o}_2^\times \rightarrow \mathbf{C}^\times$, a representation is a subrepresentation of a geometrically induced representation if and only if all its one-dimensional twists are.

In our situation, that is, for $n = 4$ and $k = 2$, there are three distinct geometric stabilizers, $P_{1,3}$, $P_{2,2}$, and $P_{3,1}$ with “unipotent radicals” $U_{1,3}$, $U_{2,2}$, and $U_{3,1}$, respectively. Thus a representation is a subrepresentation of a geometrically induced representation if and only if it is a component of $\text{Ind}_{U_{i,j}}^G \mathbf{1}$, for some $(i, j) \in \{(1, 3), (2, 2), (3, 1)\}$. Furthermore, there are three partitions, written in descending order, which embed in 2^4 and give rise to non-trivial infinitesimal induction functors, namely

$$(2, 1^3), (2^2, 1^2), (2^3, 1).$$

Thus a representation is a subrepresentation of an infinitesimally induced representation if and only if it is a component of $\text{Ind}_{U_{\lambda \hookrightarrow 2^4}}^G \mathbf{1}$, for some partition λ as above. Because of the inclusions

$$U_{(2,1^3) \hookrightarrow 2^4} \subset U_{(2^2,1^2) \hookrightarrow 2^4} \subset U_{(2^3,1) \hookrightarrow 2^4},$$

an irreducible representation of G is a component of an infinitesimally induced representation if and only if it is a component of $\text{Ind}_{U_{(2,1^3) \hookrightarrow 2^4}}^G \mathbf{1}$.

Lemma 8.1. *Suppose that π is an irreducible representation of G whose orbit contains either β_1 or β_2 . Then π is not an irreducible component of any representation geometrically induced from $P_{1,3}$ or $P_{3,1}$. Moreover, no 1-dimensional twist of π is an irreducible component of an infinitesimally induced representation.*

Proof. If π were a component of $\text{Ind}_{U_{1,3}}^G \mathbf{1}$, then $\langle \pi|_{U_{1,3}}, \mathbf{1} \rangle \neq 0$, so in particular $\langle \pi|_{N_1 \cap U_{1,3}}, \mathbf{1} \rangle \neq 0$, which implies that $\pi|_{N_1}$ contains a character ψ_b , where $b = (b_{ij})$ is a matrix such that $b_{i1} = 0$ for $i = 2, 3, 4$. This means that the characteristic polynomial of b would have a linear factor, which contradicts the hypothesis. The case of $U_{3,1}$ is treated in exactly the same way, except that the matrix b will have $b_{4j} = 0$ for $j = 1, 2, 3$. The case of infinitesimal induction is treated using the same kind of argument. Namely, if π were a component of $\text{Ind}_{U_{(2,1^3) \hookrightarrow 2^4}}^G \mathbf{1}$, then $U_{(2,1^3) \hookrightarrow 2^4} \subset N_1$ and $\langle \pi|_{U_{(2,1^3) \hookrightarrow 2^4}}, \mathbf{1} \rangle \neq 0$, which implies that $\pi|_{N_1}$ contains a character ψ_b , where $b = (b_{ij})$ is a matrix such that $b_{1j} = 0$ for $j = 1, \dots, 4$. A 1-dimensional twist of π would then contain a character ψ_{aI+b} , where a is a scalar and I is the identity matrix. The matrix $aI + b$ has a linear factor in its characteristic polynomial, which contradicts the hypothesis. \square

We now consider in order representations whose orbits contain β_1 or β_2 , respectively. In the following we will write $\bar{P}_{2,2}$ and $\bar{U}_{2,2}$ for the images mod \mathfrak{p} of the groups $P_{2,2}$ and $U_{2,2}$, respectively.

8.1. The regular cuspidal representations. Assume that π is an irreducible representation of G whose orbit contains β_1 . Since β_1 is a regular element, the representation π can be constructed explicitly as an induced representation (cf. [Hil95b]). In particular, it is shown in [Hil95b] that there exists a 1-dimensional representation ρ of $G(\psi_{\beta_1})$ (uniquely determined by π) such that $\rho|_{N_1} = \psi_{\beta_1}$, and such that

$$\pi = \text{Ind}_{G(\psi_{\beta_1})}^G \rho.$$

Proposition 8.1. *The representation π is cuspidal if and only if ρ does not contain the trivial representation of $G(\psi_{\beta_1}) \cap U_{2,2}$.*

Proof. Lemma 8.1 shows that π is cuspidal if and only if it is not a component of $\text{Ind}_{U_{2,2}}^G \mathbf{1}$. By Mackey's intertwining number theorem (cf. [CR62], 44.5), we have

$$\langle \pi, \text{Ind}_{U_{2,2}}^G \mathbf{1} \rangle = \langle \text{Ind}_{G(\psi_{\beta_1})}^G \rho, \text{Ind}_{U_{2,2}}^G \mathbf{1} \rangle = \sum_{x \in G(\psi_{\beta_1}) \backslash G/U_{2,2}} \langle \rho|_{G(\psi_{\beta_1}) \cap {}^x U_{2,2}}, \mathbf{1} \rangle,$$

so this number is zero if and only if $\langle \rho|_{G(\psi_{\beta_1}) \cap {}^x U_{2,2}}, \mathbf{1} \rangle = 0$ for each $x \in G$. Assume that π is cuspidal. Then in particular, taking $x = 1$, we have $\langle \rho|_{G(\psi_{\beta_1}) \cap U_{2,2}}, \mathbf{1} \rangle = 0$.

Conversely, assume that π is not cuspidal. Then $\langle \rho|_{G(\psi_{\beta_1}) \cap {}^x U_{2,2}}, \mathbf{1} \rangle \neq 0$, for some $x \in G$, and in particular, $\langle \rho|_{N_1 \cap {}^x U_{2,2}}, \mathbf{1} \rangle = \langle \psi_{\beta_1}|_{N_1 \cap {}^x U_{2,2}}, \mathbf{1} \rangle \neq 0$. Write \bar{x} for x modulo \mathfrak{p} . Now $\psi_{\beta_1}|_{N_1 \cap {}^x U_{2,2}} = \psi_{\beta_1}|_{x(N_1 \cap U_{2,2})}$, and $\psi_{\beta_1}(^x g) = \psi_{\bar{x}^{-1}\beta_1\bar{x}}(g)$, for any $g \in N_1 \cap U_{2,2}$. Let $\bar{x}^{-1}\beta_1\bar{x}$ be represented by the matrix

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where each A_{ij} is a 2×2 -block. Then from the definition of $\psi_{\bar{x}^{-1}\beta_1\bar{x}}$ and the condition $\psi_{\bar{x}^{-1}\beta_1\bar{x}}(g) = 1$, for all $g \in N_1 \cap U_{2,2}$, it follows that $A_{21} = 0$; thus

$$\bar{x}^{-1}\beta_1\bar{x} \in \bar{P}_{2,2}.$$

Since $\bar{x}^{-1}\beta_1\bar{x}$ is a block upper-triangular matrix with the same characteristic polynomial as β_1 , we must have $A_{11} = B_1\eta B_1^{-1}$, $A_{22} = B_2\eta B_2^{-1}$, for some $B_1, B_2 \in \text{GL}_2(\mathbf{F}_q)$. Then there

exists $p \in \bar{P}_{2,2}$ such that

$$(\bar{x}p)^{-1}\beta_1(\bar{x}p) = \begin{pmatrix} \eta & B \\ 0 & \eta \end{pmatrix},$$

for some $B \in M_2(\mathbf{F}_q)$ (in fact, we can take $p = \begin{pmatrix} B_1^{-1} & 0 \\ 0 & B_2^{-1} \end{pmatrix}$). The Levi decomposition $\bar{P}_{2,2} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ (written in block matrix form) applied to β_1 and $(\bar{x}p)^{-1}\beta_1(\bar{x}p)$ implies that the semisimple parts $(\bar{x}p)^{-1} \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix} (\bar{x}p)$ and $\begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix}$ are equal, that is,

$$\bar{x}p \in C_{G_{14}}\left(\begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix}\right) = G(\beta_2) \cong \mathrm{GL}_2(\mathbf{F}_{q^2}).$$

Now, in $G(\beta_2)$, the equation $(\bar{x}p)^{-1}\beta_1(\bar{x}p) = \begin{pmatrix} \eta & B \\ 0 & \eta \end{pmatrix}$ implies that $\bar{x}p \in \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \cap G(\beta_2) \subset \bar{P}_{2,2}$, so $\bar{x} \in \bar{P}_{2,2}$, and hence $x \in N_1 P_{2,2}$. The facts that $U_{2,2}$ is normal in $P_{2,2}$, and that $\langle \rho|_{G(\psi_{\beta_1}) \cap {}^x U_{2,2}}, \mathbf{1} \rangle$ only depends on the right coset of x modulo N_1 then imply that

$$0 \neq \langle \rho|_{G(\psi_{\beta_1}) \cap {}^x U_{2,2}}, \mathbf{1} \rangle = \langle \rho|_{G(\psi_{\beta_1}) \cap U_{2,2}}, \mathbf{1} \rangle.$$

□

The preceding proposition shows that we can construct all the cuspidal representations of G with orbit containing β_1 by constructing the corresponding ρ on $G(\psi_{\beta_1})$. Since ψ_{β_1} is trivial on $N_1 \cap U_{2,2}$, we can extend ψ_{β_1} to a representation of $(G(\psi_{\beta_1}) \cap U_{2,2})N_1$, trivial on $G(\psi_{\beta_1}) \cap U_{2,2}$. Then ψ_{β_1} can be extended to a representation $\tilde{\psi}_{\beta_1}$ on the whole of $G(\psi_{\beta_1})$, such that $\tilde{\psi}_{\beta_1}$ is trivial on $G(\psi_{\beta_1}) \cap U_{2,2}$ (this incidentally shows that there exist irreducible non-cuspidal representations of G whose orbit contains β_1). Now let θ be a representation of $G(\psi_{\beta_1})$ obtained by pulling back a representation of $G(\psi_{\beta_1})/N_1$ that is non-trivial on $(G(\psi_{\beta_1}) \cap U_{2,2})N_1/N_1$. Then $\rho := \theta \otimes \tilde{\psi}_{\beta_1}$ is a representation of $G(\psi_{\beta_1})$ which is a lift of ψ_{β_1} , and which is non-trivial on $G(\psi_{\beta_1}) \cap U_{2,2}$. By a standard fact in representation theory, all the lifts of ψ_{β_1} to $G(\psi_{\beta_1})$ are of the form $\theta \otimes \tilde{\psi}_{\beta_1}$ for some θ trivial on N_1 . Thus all the representations of $G(\psi_{\beta_1})$ which are lifts of ψ_{β_1} and which are non-trivial on $G(\psi_{\beta_1}) \cap U_{2,2}$, are of the form above, namely $\theta \otimes \tilde{\psi}_{\beta_1}$ where θ is trivial on N_1 but non-trivial on $G(\psi_{\beta_1}) \cap U_{2,2}$. We note that in the regular case, distinct representations θ give rise to distinct lifts $\theta \otimes \tilde{\psi}_{\beta_1}$. This can be seen by a counting argument, in the following way. Because β_1 lies in a regular orbit, we can write $G(\psi_{\beta_1}) = C_G(\hat{\beta}_1)N_1$, for some element $\hat{\beta}_1 \in M_2(\mathfrak{o}_2)$ with image $\beta_1 \bmod \mathfrak{p}$. Then because $C_G(\hat{\beta}_1)$ is abelian, there are exactly $(C_G(\hat{\beta}_1) : C_G(\hat{\beta}_1) \cap N_1) = |G(\psi_{\beta_1})/N_1|$ characters χ of $C_G(\hat{\beta}_1)$ which agree with ψ_{β_1} on $C_G(\hat{\beta}_1) \cap N_1$, and each of them gives rise to a representation $\chi\psi_{\beta_1}$ of $G(\psi_{\beta_1})$ defined by $\chi\psi_{\beta_1}(cn) = \chi(c)\psi_{\beta_1}(n)$, for $c \in C_G(\hat{\beta}_1)$, $n \in N_1$. Clearly every lift of ψ_{β_1} to $G(\psi_{\beta_1})$ must be equal to some such χ on $C_G(\hat{\beta}_1)$, and distinct χ give rise to distinct representations $\chi\psi_{\beta_1}$. Since the number of lifts of ψ_{β_1} to $G(\psi_{\beta_1})$ is thus equal to the number of representations of $G(\psi_{\beta_1})/N_1$, we see that distinct θ give rise to distinct representations $\theta \otimes \tilde{\psi}_{\beta_1}$. Now by a standard result in Clifford theory, distinct irreducible representations of $G(\psi_{\beta_1})$ containing ψ_{β_1} (when restricted to N_1) induce to distinct irreducible representations of G . Thus, distinct representations θ give rise to distinct representations $\mathrm{Ind}_{G(\psi_{\beta_1})}^G \rho$, although the correspondence $\theta \mapsto \mathrm{Ind}_{G(\psi_{\beta_1})}^G \rho$ is by no means canonical, due to the choice of $\tilde{\psi}_{\beta_1}$. Similarly, if we are considering the lifts $\chi\psi_{\beta_1}$, then the construction depends on the choice of $\hat{\beta}_1$.

The above parameterizations of representations of $G(\psi_{\beta_1})$ containing ψ_{β_1} , both involve non-canonical choices, although the set of representations obtained is certainly uniquely determined. Nevertheless, Proposition 8.1 shows that there is a canonical 1-1 correspondence (given simply by induction) between on the one hand irreducible representations of $G(\psi_{\beta_1})$ which contain ψ_{β_1} and which are non-trivial on $G(\psi_{\beta_1}) \cap U_{2,2}$, and on the other hand cuspidal representations of G with β_1 in their respective orbits. We shall now extend this result to cuspidal representations which have β_2 in their respective orbits, and thus cover all cuspidal representations of G .

8.2. The irregular cuspidal representations. Assume now that π is an irreducible representation of G whose orbit contains β_2 . Although β_2 is not regular, it is strongly semisimple in the sense of [Hil95a], Definition 3.1, and thus π can be constructed explicitly in a way similar to the regular case. More precisely, Proposition 3.3 in [Hil95a] implies that there exists an irreducible representation $\tilde{\psi}_{\beta_2}$ of $G(\psi_{\beta_2})$, such that $\tilde{\psi}_{\beta_2}|_{N_1} = \psi_{\beta_2}$, and any extension of ψ_{β_2} to $G(\psi_{\beta_2})$ is of the form $\rho := \theta \otimes \tilde{\psi}_{\beta_2}$, for some irreducible representation θ pulled back from a representation of $G(\psi_{\beta_2})/N_1$. Then

$$\pi = \text{Ind}_{G(\psi_{\beta_2})}^G \rho$$

is an irreducible representation, any representation of G with β_2 in its orbit is of this form, and as in the regular case, ρ is uniquely determined by π . We then have a result completely analogous to the previous proposition:

Proposition 8.2. *The representation π is cuspidal if and only if ρ does not contain the trivial representation of $G(\psi_{\beta_2}) \cap U_{2,2}$.*

Proof. The proof of Proposition 8.1 with β_1 replaced by β_2 , goes through up to the point where (under the assumption that π is not cuspidal) we get $\bar{x}p \in C_{G_{14}}\left(\begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix}\right) = G(\psi_{\beta_2})/N_1$. It then follows that $x \in G(\psi_{\beta_2})P_{2,2}$, and since $U_{2,2}$ is normal in $P_{2,2}$, and $\langle \rho|_{G(\psi_{\beta_2}) \cap {}^x U_{2,2}}, \mathbf{1} \rangle$ only depends on the right coset of x modulo $G(\psi_{\beta_2})$, we get

$$0 \neq \langle \rho|_{G(\psi_{\beta_2}) \cap {}^x U_{2,2}}, \mathbf{1} \rangle = \langle \rho|_{G(\psi_{\beta_2}) \cap U_{2,2}}, \mathbf{1} \rangle.$$

□

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